

# A bijective study of Basketball walks

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## Abstract

The Catalan numbers count many classes of combinatorial objects. The most emblematic such objects are probably the Dyck walks and the binary trees, and, whenever another class of combinatorial objects is counted by the Catalan numbers, it is natural to search for an explicit bijection between the latter objects and one of the former objects. In most cases, such a bijection happens to be relatively simple but it might sometimes be more intricate.

In this work, we focus on so-called *basketball walks*, which are integer-valued walks with step-set  $\{-2, -1, +1, +2\}$ . The presence of  $-2$  as an allowed step makes it impossible to use the classical Łukasiewicz encoding of trees by integer-valued walks, and thus a different strategy is needed. We give an explicit bijection that maps, for each  $n \geq 2$ ,  $n$ -step basketball walks from 0 to 0 that visit 1 and are positive except at their extremities to  $n$ -leaf binary trees. Moreover, we can partition the steps of a walk into  $\pm 1$ -steps, odd  $+2$ -steps or even  $-2$ -steps, and odd  $-2$ -steps or even  $+2$ -steps, and these three types of steps are mapped through our bijection to double leaves, left leaves, and right leaves of the corresponding tree.

We also prove that basketball walks from 0 to 1 that are positive except at the origin are in bijection with increasing unary-binary trees with associated permutation avoiding 213. We furthermore give the refined generating function of these objects with an extra variable accounting for the unary nodes.

# 1 Introduction

An integer-valued *walk* is a finite sequence of integers, usually starting at 0. Its *steps* are the differences of two consecutive values it takes. Walks with a given possible step-set have been the focus of many studies [Ges80, LY90, Duc00, BF02, BM08, Kra15] and so-called Łukasiewicz walks (walks whose step-set  $\mathcal{S}$  satisfies  $-1 \in \mathcal{S} \subseteq \{-1, 0, 1, 2, \dots\}$ ) are of particular interest as they encode plane trees whose vertex outdegrees all lie in  $\mathcal{S} + 1$  [Lot97, Chapter 11].

In this work, we consider the simplest case of walks that are not Łukasiewicz walks, namely *basketball walks*. These are walks with step-set  $\{-2, -1, +1, +2\}$ ; they were named by Ayer and

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Zeilberger [AZ07] from the fact that they record the score difference between two teams playing a basketball game at a time where three-pointers did not exist. In the aforementioned paper, motivated by their connections with constrained polymers problems, the authors are more precisely interested in the generating function of basketball walks from 0 to 0 that remain inside a set of the form  $\{0, 1, \dots, w\}$  at all times. Basketball walks were also later considered by Bousquet-Mélou [BM08], who looked at the structure of the algebraic equations satisfied by the generating function of excursions, as well as that of bounded excursions. The latter has also been studied recently [Bac13].

More recently, using the so-called kernel method, Banderier, Krattenthaler, Krinik, Kruchinin, Kruchinin, Nguyen and Wallner [BKK<sup>+</sup>16] showed that the generating function of basketball walks from 0 to 1 that are positive except at the origin, counted with weight  $z$  per step, is given<sup>1</sup> by

$$G(z) = -\frac{1}{2} + \frac{1}{2} \sqrt{\frac{2 - 3z - 2\sqrt{1 - 4z}}{z}}.$$

They also derived summatory expressions for the number of such walks of given length, and for the number of basketball excursions of a given length, as well as asymptotics for these numbers. Finally, they noted, and this was the starting point of the present work, that the generating function  $G$  is related to the Catalan generating function  $C$ , characterized by  $C(z) = 1 + zC(z)^2$ , by the simple equation [BKK<sup>+</sup>16, Equation (3.14)]:

$$1 + G(z) + G^2(z) = C(z). \quad (1)$$

Instead of considering  $G$ -walks, we will rather focus on basketball walks from 0 to 0 that visit 1 and are positive except at the extremities: we call them  $U$ -walks and denote by  $U$  their generating function. A first step decomposition of  $U$ -walks yields that  $U(z) = zG(z) + zG^2(z)$  as a  $U$ -walk is either a  $+1$ -step followed by a reversed  $G$ -walk or a  $+2$ -step followed by a reversed  $G$ -walk and another reversed  $G$ -walk (recall that  $U$ -walks have to visit 1). Expressed in terms of  $U$ , Equation (1) becomes

$$U(z) = z(C(z) - 1), \quad (2)$$

which is the generating function of nontrivial binary trees counted with weight  $z$  per leaf. This is the equation to which we give a bijective interpretation. As a byproduct of our bijection, we obtain the following proposition. We say that a step of a walk is *even* (resp. *odd*) if it starts at even (resp. odd) height.

**Proposition 1.** *The number of basketball walks from 0 to 0 that visit 1 and are positive except at the extremities, with  $2d \pm 1$ -steps,  $\ell$  odd  $+2$ -steps or even  $-2$ -steps, and  $r$  odd  $-2$ -steps or even  $+2$ -steps is equal to*

$$\frac{1}{d} \binom{2d-2}{d-1} \binom{\ell+r+2d-2}{\ell+r} \binom{\ell+r}{\ell}.$$

In the second half of this article, we show how basketball walks are related to increasing unary-binary trees with associated permutation avoiding 213. Recall that a *unary-binary tree* of size  $n \geq 1$  is a plane tree with  $n$  vertices, each having 0, 1 or 2 children. It is *increasing* if its vertices are

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<sup>1</sup>In [BKK<sup>+</sup>16], the authors use the notation  $G_{0,1}$ . We removed the subscript in order to lighten the notation.

bijectively labeled with  $1, \dots, n$  in such a way that any vertex receives a label larger than its parent's. In other words, the sequence of labels read along any branch of the tree from the root to a leaf is increasing. With an increasing tree, we associate the permutation obtained by reading the labels of the tree in breadth-first search order, from left to right. Recall that a permutation  $\sigma = (\sigma_1, \dots, \sigma_n)$  is said to *avoid* the pattern  $\pi = (\pi_1, \dots, \pi_k)$  if, for any  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ , the permutation obtained from  $(\sigma_{i_1}, \dots, \sigma_{i_k})$  after relabeling with  $1, \dots, k$  in the same relative order is different from  $\pi$ . (Note that, in particular, a permutation of size  $n$  avoids any pattern of size strictly larger than  $n$ .)

The study of increasing trees whose associated permutation avoids a given pattern was initiated by Riehl (see for instance [LPRS16]). By computer programming, she observed that the first terms of the sequence of increasing unary-binary trees with associated permutation avoiding 213 coincide with those of the sequence referenced A166135 in the On-Line Encyclopedia of Integer Sequences, which was later proved in [BKK<sup>+</sup>16] to be the sequence with generating function  $G$ . We show in the present work that this is indeed the case and thus answer [BKK<sup>+</sup>16, Conjecture 5.2].

**Theorem 2.** *For any  $n \geq 0$ , the number of  $n$ -step basketball walks from 0 to 1 that are positive except at the origin is equal to the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213.*

We will furthermore explain how to obtain an explicit bijection between the classes of objects appearing in Theorem 2. It is well known that permutations avoiding 213 are yet another class of combinatorial objects counted by Catalan numbers. We are thus interested in a family of trees carrying an extra combinatorial structure counted by Catalan. Of course, both structures are strongly dependent because of the condition of being increasing. We will however see in Lemma 7 that the two structures can be separated and the conditions the permutation has to fulfill are completely encoded in the sequence of bits obtained by reading the tree in breadth-first search order and recording whether the vertices encountered are nodes or leaves<sup>2</sup>.

As an immediate consequence of this theorem, for  $n \geq 1$ , the following expressions found in [BKK<sup>+</sup>16, Proposition 3.5]

$$[z^n]G(z) = \frac{1}{n} \sum_{k=1}^n (-1)^{k+1} \binom{2k-2}{k-1} \binom{2n}{n-k} = \frac{1}{n} \sum_{i=0}^n \binom{n}{i} \binom{n}{2n+1-3i}$$

also give the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213. Moreover, [BKK<sup>+</sup>16, Theorem 3.8] gives an asymptotic estimate of this number.

Finally, we study a refinement of the previous formula, which takes as an extra parameter the number of unary nodes, that is, nodes with only one child.

**Proposition 3.** *For  $n \geq 1$  and  $0 \leq k \leq \lfloor (n-1)/2 \rfloor$ , the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 that have exactly  $n-1-2k$  unary nodes is equal to*

$$\frac{1}{n} \binom{2n}{k} \binom{n-k}{k+1}.$$

*In particular, we obtain yet another expression for  $[z^n]G(z)$ :*

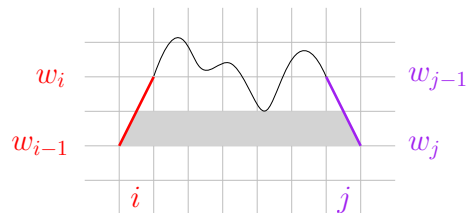
$$[z^n]G(z) = \frac{1}{n} \sum_{k \geq 0} \binom{2n}{k} \binom{n-k}{k+1}.$$

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<sup>2</sup>Recall that a vertex is called a *node* if it has at least one child and a *leaf* otherwise.

In the light of Theorem 2, it is natural to try and find the statistics corresponding to unary nodes on basketball walks: it is the number of what we call *staggered  $\pm 2$ -steps* (see Figure 1).

**Definition 1.** Let  $(w_0, w_1, \dots, w_n)$  be the successive heights taken by a basketball walk and let, for all integer  $1 \leq i \leq n$ ,  $u_i := w_i - w_{i-1}$  be the value of its  $i$ -th step. For  $i < j$ , we say that  $u_i$  and  $u_j$  are paired if  $u_i = +2$ ,  $u_j = -2$ ,  $w_j = w_{i-1}$ , and for all  $i < k < j$ , we have  $w_k \geq w_{i-1} + 1$ . A staggered  $\pm 2$ -step is a  $\pm 2$ -step that is not paired with any other  $\pm 2$ -step.



**Figure 1:** The  $i$ -th and  $j$ -th steps of this walk are paired (the open gray area must not be intersected by the walk). A staggered  $\pm 2$ -step is a  $\pm 2$ -step that is not paired with any other  $\pm 2$ -step.

**Proposition 4.** For  $n \geq 1$  and  $m \geq 0$ , the number of  $n$ -step basketball walks from 0 to 1 that are positive except at the origin and that contain exactly  $m$  staggered  $\pm 2$ -steps is equal to the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 that have exactly  $m$  unary nodes.

Proposition 3 gives a closed formula for these numbers. Along the same lines, we obtain a similar expression for basketball excursions instead of  $G$ -walks.

**Proposition 5.** For  $n \geq 2$ , the number of  $(n - 1)$ -step nonnegative basketball walks from 0 to 0 is equal to

$$\frac{1}{n} \sum_{k \geq 0} \binom{2n}{k} \binom{n - k - 2}{k - 1}$$

and, for  $k \geq 0$ , the term of index  $k$  in the previous sum is the number of walks with exactly  $n - 1 - 2k$  staggered  $\pm 2$ -steps.

The remaining of the paper is organized into two sections. In section 2, we present and study the bijection between  $U$ -walks and binary trees. In Section 3, we investigate the link between  $G$ -walks and increasing unary-binary trees with associated permutation avoiding 213.

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## 2 Basketball walks and binary trees

### 2.1 Generalities

Throughout this paper, we will denote generating functions by capital letters. The class of combinatorial objects counted by a generating function will be denoted by the same letter with a cal-

ligraphic font. For instance, the generating function  $A$  will count the objects of the combinatorial class  $\mathcal{A}$ . Moreover, we denote by  $\mathbf{1}$  (resp.  $\mathcal{Z}$ ) the combinatorial class whose only element has size 0 (resp. size 1), that is, the combinatorial class corresponding to the generating function  $z \mapsto 1$  (resp.  $z \mapsto z$ ). Finally, for a class  $\mathcal{X}$  with no object of size 0, we denote by  $\text{Seq}(\mathcal{X})$  the combinatorial class of sequences of elements of  $\mathcal{X}$ , that is  $\mathbf{1} + \mathcal{X} + \mathcal{X}^2 + \dots$ , and by  $\text{Seq}_{\geq k}(\mathcal{X}) = \mathcal{X}^k \text{Seq}(\mathcal{X})$  the class of sequences of at least  $k$  elements of  $\mathcal{X}$ .

Let us start by introducing the following combinatorial classes of basketball walks (recall that this means walks with steps in  $\{-2, -1, +1, +2\}$ ):

- ◊  $\mathcal{U}$  is the set of walks from 0 to 0 that visit 1 and are positive except at the extremities;
- ◊  $\mathcal{A}$  is the set of nonnegative walks from 0 to 0;
- ◊  $\mathcal{B}$  is the set of walks from 0 to 1 that visit 0 and 1 only at the extremities.

The corresponding generating functions  $U$ ,  $A$  and  $B$  count walks of these classes with a weight  $z$  per step. We also call  $X$ -walk an element of a class  $\mathcal{X}$  of walks.

**Notation.** It will sometimes be convenient to see walks as sequences of steps. Note that there is a slight abuse in such a notation, as we forget the value of the origin of the walks; anywhere we use this notation, the initial value of the walk will be implicit or irrelevant. We will denote by  $\bar{\mathbf{a}} := (-a_n, \dots, -a_1)$  the reverse of a walk  $\mathbf{a} = (a_1, \dots, a_n)$  and by  $\mathbf{ab} := (a_1, \dots, a_n, b_1, \dots, b_m)$  the concatenation of  $\mathbf{a}$  and  $\mathbf{b} = (b_1, \dots, b_m)$ .

We can notice right away that the class  $\mathcal{G}$  of basketball walks from 0 to 1 that are positive except at the origin, introduced in Section 1, satisfies  $\mathcal{G} = \mathcal{B}\mathcal{A}$  by splitting any  $G$ -walk at the first time it reaches height 1. Moreover, we can see that  $\mathcal{B} = \mathcal{Z} \text{Seq}(\mathcal{Z}\mathcal{A})$  as follows (see Figure 2b). A  $B$ -walk is either trivial, that is, reduced to a single  $+1$ -step, or starts with a  $+2$ -step, forms an  $A$ -walk between the first and last time it visits 2, and then forms a reversed  $B$ -walk from its last visit of 2 to its end. As a result,  $\mathcal{B} = \mathcal{Z} + \mathcal{Z}\mathcal{A}\mathcal{B}$ , and the claim follows. This implies that

$$\mathcal{G} = \mathcal{Z}\mathcal{A} + \mathcal{Z}\mathcal{A}\mathcal{G}, \quad \text{so that} \quad \mathcal{G} = \text{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A}). \quad (3)$$

In the remaining of Section 2, we will focus on  $\mathcal{U}$  and no longer on  $\mathcal{G}$ . We will come back to  $\mathcal{G}$  and use (3) in Section 3. Decomposing  $A$ -walks, we obtain the following identity (see Figure 2a)

$$\mathcal{A} = \mathbf{1} + \mathcal{Z}\mathcal{A}\mathcal{Z}\mathcal{A} + \mathcal{B}\mathcal{A}\mathcal{B}\mathcal{A}, \quad (4)$$

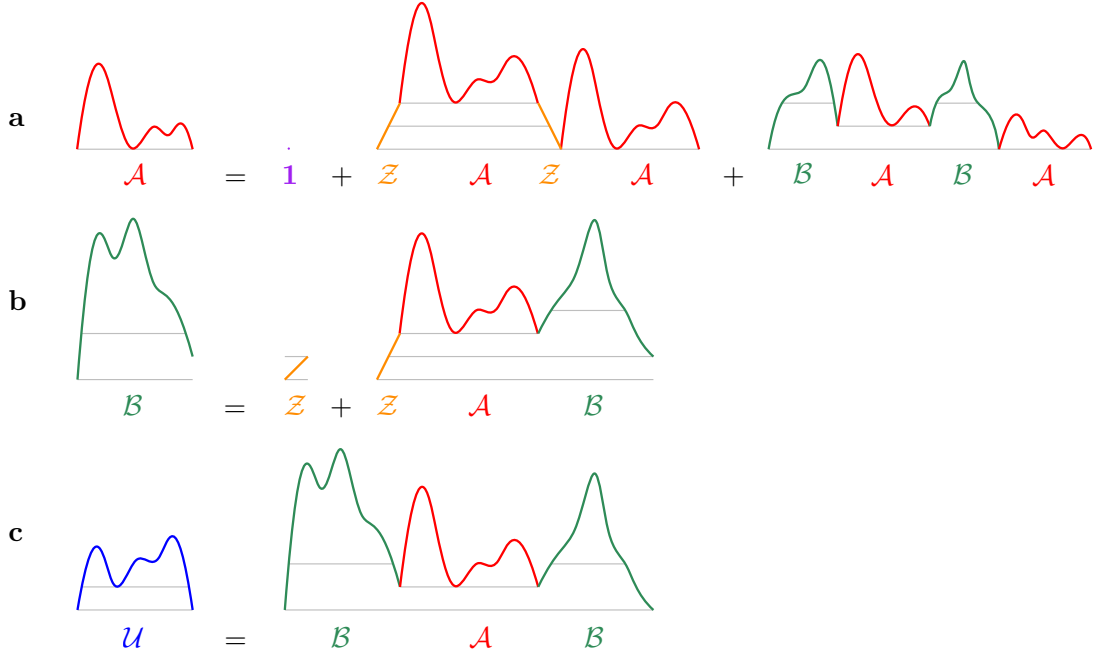
which corresponds to the fact that an  $A$ -walk is either empty, or does not visit 1 before its first return to 0, or does visit 1 before its first return to 0.

Splitting a  $U$ -walk at the first and last time it visits 1, we see that  $\mathcal{U} = \mathcal{B}\mathcal{A}\mathcal{B}$  (see Figure 2c). We claim that the previous identities suffice to show that  $\mathcal{U}$  satisfies  $\mathcal{U} = (\mathcal{Z} + \mathcal{U})^2$  and thus is the Catalan class whose generating function is given by (2). Indeed, if we multiply (4) by  $\mathcal{B}^2$ , we obtain

$$\mathcal{B}\mathcal{A}\mathcal{B} = \mathcal{B}^2 + (\mathcal{Z}\mathcal{A}\mathcal{B})^2 + (\mathcal{B}\mathcal{A}\mathcal{B})^2 \quad \text{and thus} \quad \mathcal{U} = \mathcal{B}^2 + (\mathcal{Z}\mathcal{A}\mathcal{B})^2 + \mathcal{U}^2.$$

Now observe that, for an arbitrary combinatorial class  $\mathcal{R}$  with no object of size 0, we have

$$\begin{aligned} \text{Seq}(\mathcal{R})^2 + (\mathcal{R} \text{Seq}(\mathcal{R}))^2 &= \mathbf{1}.\mathbf{1} + \mathbf{1}.\text{Seq}_{\geq 1}(\mathcal{R}) + \text{Seq}_{\geq 1}(\mathcal{R}) \text{Seq}(\mathcal{R}) + \text{Seq}_{\geq 1}(\mathcal{R}) \text{Seq}_{\geq 1}(\mathcal{R}) \\ &= \mathbf{1} + 2 \text{Seq}_{\geq 1}(\mathcal{R}) \text{Seq}(\mathcal{R}) = \mathbf{1} + 2\mathcal{R} \text{Seq}(\mathcal{R})^2. \end{aligned}$$



**Figure 2:** Decomposition of an  $A$ -walk (a), a  $B$ -walk (b) and a  $U$ -walk (c). The horizontal lines are here to emphasize the facts that  $A$ -walks are nonnegative,  $B$ -walks only visit 0 and 1 at their extremities and  $U$ -walks only visit 0 at their extremities. Beware, however, that the walk  $(+1)$  is a  $B$ -walk.

To obtain the first equality, we split  $\text{Seq}(\mathcal{R})^2$  into three terms and we recombined the second and forth terms in order to obtain the second equality. As  $\mathcal{B} = \mathcal{Z} \text{Seq}(\mathcal{Z}\mathcal{A})$ , the latter identity applied to  $\mathcal{R} = \mathcal{Z}\mathcal{A}$  yields

$$\mathcal{B}^2 + (\mathcal{Z}\mathcal{A}\mathcal{B})^2 = \mathcal{Z}^2(1 + 2\mathcal{Z}\mathcal{A}\text{Seq}(\mathcal{Z}\mathcal{A})^2) = \mathcal{Z}^2 + 2\mathcal{Z}\mathcal{A}\mathcal{B}^2 = \mathcal{Z}^2 + 2\mathcal{Z}\mathcal{U},$$

so that we obtain  $\mathcal{U} = \mathcal{Z}^2 + 2\mathcal{Z}\mathcal{U} + \mathcal{U}^2 = (\mathcal{Z} + \mathcal{U})^2$  as claimed.

If we now mark each  $\pm 1$ -step by  $\mathcal{Z}_1$  and each  $\pm 2$ -step by  $\mathcal{Z}_2$ , then it is easy to see that the identities above become

$$\mathcal{A} = 1 + \mathcal{Z}_2\mathcal{A}\mathcal{Z}_2\mathcal{A} + \mathcal{B}\mathcal{A}\mathcal{B}, \quad \mathcal{B} = \mathcal{Z}_1 \text{Seq}(\mathcal{Z}_2\mathcal{A}), \quad \mathcal{U} = \mathcal{B}\mathcal{A}\mathcal{B},$$

so that  $\mathcal{U}$  now satisfies

$$\mathcal{U} = \mathcal{B}^2 + (\mathcal{Z}_2\mathcal{A}\mathcal{B})^2 + \mathcal{U}^2 = \mathcal{Z}_1^2 + 2\mathcal{Z}_2\mathcal{U} + \mathcal{U}^2.$$

In a binary tree, we define a *double leaf* to be a leaf whose sibling is also a leaf and a *simple leaf* to be a leaf whose sibling is a node. The previous identity corresponds to the decomposition grammar of binary trees where each double leaf is marked by  $\mathcal{Z}_1$  and each simple leaf is marked by  $\mathcal{Z}_2$ .

Note that a simple leaf is either the left child or the right child of its parent; it is called a *left leaf* of a *right leaf* accordingly. In the next section, we present a recursive bijection between  $\mathcal{U}$

and binary trees that essentially implements the above decomposition, (thus mapping  $\pm 1$ -steps to double leaves and  $\pm 2$ -steps to simple leaves), along with suitable reversal operations in order to further map the two types of simple leaves (left or right) of the binary tree to explicit (parity-dependent) types of  $\pm 2$ -steps on the corresponding  $U$ -walk.

## 2.2 The bijection

We will now rewrite the grammar of the basketball walks we consider in a way that reflects more faithfully the Catalan decomposition. Recall from Figure 2c that  $\mathcal{U} = \mathcal{B}A\mathcal{B}$ . This decomposition will be highly used in what follows.

As a  $U$ -walk starts and end at 0, is positive inbetween and has steps of height at most 2, it makes sense to focus on the sequence of heights it reaches, restricted to the set  $\{1, 2\}$ . For instance, the sequence corresponding to the walk taking the consecutive values  $(0, 2, 3, 4, 2, 3, 1, 2, 3, 5, 4, 2, 3, 1, 3, 1, 2, 0)$  is  $(2, 2, 1, 2, 2, 1, 1, 2)$ . We classify  $U$ -walks into 4 classes according to this sequence of values as follows (the boxed values are mandatory<sup>3</sup>, each  $x$  can be either 1 or 2). See Figure 3.

- (i)  $\boxed{1}$ ;
- (ii)  $2 \dots 2 \boxed{1} \boxed{2} 2 \dots 2 \boxed{1} x \dots x$ ;
- (iii)  $2 \dots 2 \boxed{1} \boxed{2} 2 \dots 2$ ;
- (iv)  $2 \dots 2 \boxed{2} \boxed{1}$  or  $2 \dots 2 \boxed{1} \boxed{1} x \dots x$ .

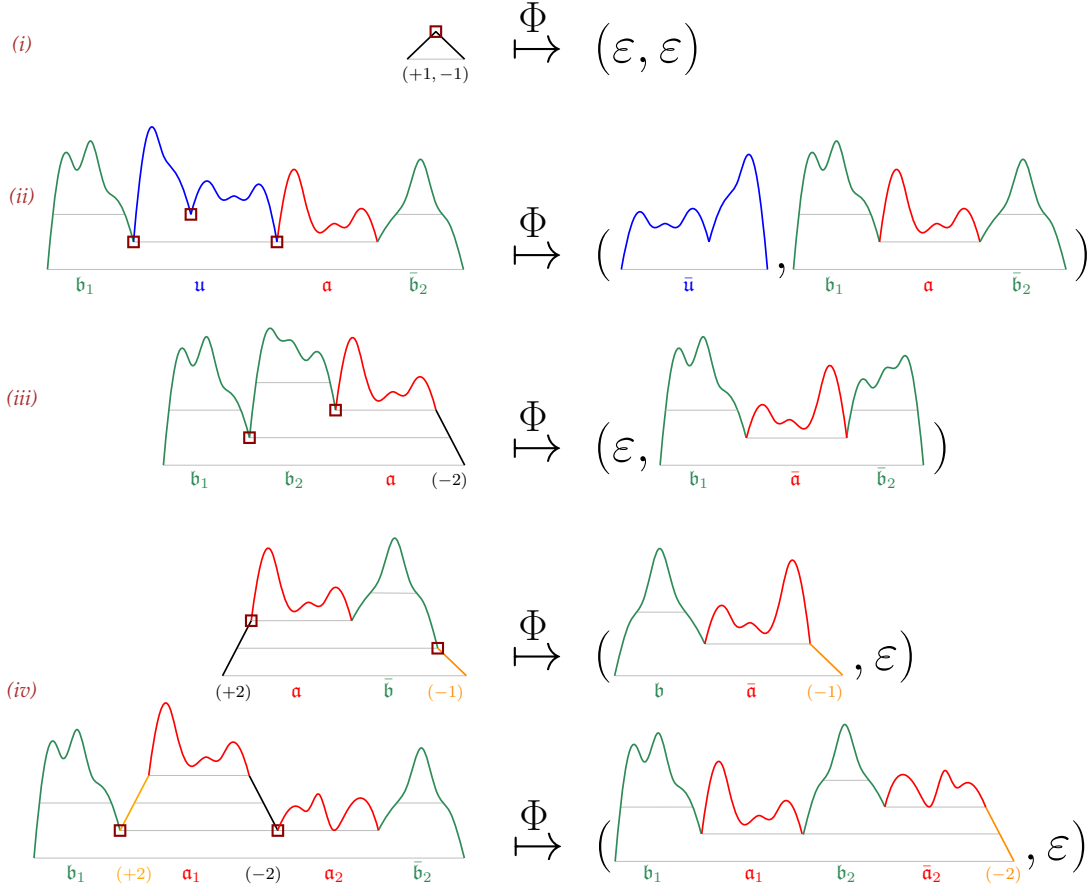
Let us analyze more closely these different classes and, at the same time, define a function  $\Phi : \mathcal{U} \rightarrow (\mathcal{U} \cup \{\varepsilon\})^2$ , where  $\varepsilon$  is a formal symbol. The function  $\Phi$  will be the starting point of the bijection. We refer the reader to Figure 3 for a visual aid.

**Class (i).** The only walk in class (i) is  $(+1, -1)$ ; we define  $\Phi((+1, -1)) := (\varepsilon, \varepsilon)$ .

**Class (ii).** Such a walk is the concatenation of three subwalks: a  $B$ -walk  $b_1$  ( $2 \dots 2 \boxed{1}$ ), a  $U$ -walk  $u$  ( $\boxed{1} \boxed{2} 2 \dots 2 \boxed{1}$ ), and the concatenation of an  $A$ -walk  $a$  and a reversed  $B$ -walk  $\bar{b}_2$  ( $\boxed{1} x \dots x$ ). We set  $\Phi(b_1 u a \bar{b}_2) := (\bar{u}, b_1 a \bar{b}_2)$ . (At this point, it might seem odd to have chosen  $\bar{u}$  instead of  $u$ ; this choice will only have bearings in Section 2.3. See the caption of Figure 3 for more details.)

**Class (iii).** Such a walk is composed of a  $B$ -walk  $b_1$  ( $2 \dots 2 \boxed{1}$ ), another  $B$ -walk  $b_2$  ( $\boxed{1} \boxed{2}$ ), an  $A$ -walk  $a$  and a  $-2$ -step ( $\boxed{2} 2 \dots 2$ ). We set  $\Phi(b_1 b_2 a (-2)) := (\varepsilon, b_1 a \bar{b}_2)$ .

<sup>3</sup>For instance, a walk in class (ii) will visit 2 a nonnegative number of times, then 1 for the first time, then 2 at least once before it visits 1 for the second time and then is free to visit 1 and 2 an arbitrary number of times and in any order.



**Figure 3:** The four different classes and the definition of the function  $\Phi$ . The reversal operation on some  $U$ -walks and  $A$ -walks was done in order to preserve the statistics of Proposition 1. For the purposes of Section 2.2, an alternate definition of  $\Phi$  without these reversal operations also works. However, the choices of reversals we made will be crucial for Section 2.3.

**Class (iv).** A walk in the first subclass is made of a 2-step and an  $A$ -walk  $a$  ( $2 \dots 2 \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ ), then a reversed  $B$ -walk  $\bar{b}$  and a  $-1$ -step ( $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ ). We set  $\Phi((+2)a\bar{b}(-1)) := (b\bar{a}(-1), \varepsilon)$ .

A walk in the second subclass can be decomposed into a  $B$ -walk  $b_1$  ( $2 \dots 2 \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ ), a  $+2$ -step, an  $A$ -walk  $a_1$  and a  $-2$ -step ( $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$ ), and the concatenation of an  $A$ -walk  $a_2$  and a reversed  $B$ -walk  $\bar{b}_2$  ( $\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} x \dots x$ ). We forget the  $-2$ -step and rearrange the remaining parts as follows: we set  $\Phi(b_1(+2)a_1(-2)a_2\bar{b}_2) := (b_1a_1b_2\bar{a}_2(-2), \varepsilon)$ .

Observe that the walks in the two previous subclasses together yield an even  $+2$ -step or an odd  $-2$ -step, as well as a  $U$ -walk whose final step is  $-1$  or  $-2$  depending on the subclass.

We recover from the previous classification that  $\mathcal{U} = \mathcal{Z}^2 + \mathcal{U}^2 + \mathcal{Z}\mathcal{U} + \mathcal{Z}\mathcal{U} = (\mathcal{Z} + \mathcal{U})^2$ . To make



the parallel with Section 2.1, we wrote

$$\mathcal{U} = \underbrace{\mathcal{Z}^2}_{(i)} + \underbrace{\mathcal{Z} \cdot \mathcal{Z} \text{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A})}_{(iv).1} + \underbrace{\mathcal{Z} \text{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A}) \cdot \mathcal{Z} \text{Seq}(\mathcal{Z}\mathcal{A})}_{(iii)} + \underbrace{\mathcal{Z}^2 \mathcal{A}^2 (\mathcal{Z} \text{Seq}(\mathcal{Z}\mathcal{A}))^2}_{(iv).2} + \underbrace{\mathcal{U}^2}_{(ii)}.$$

Now, the previous analysis can be turned into a (recursive) bijection between  $U$ -walks and binary trees. We take a  $U$ -walk and recursively build the corresponding tree as follows. We start from a vertex tree and assign to its seldom node the  $U$ -walk. Then, recursively, we do the following for every node with an assigned walk. We denote by  $(\mathfrak{w}_1, \mathfrak{w}_2)$  the value of  $\Phi$  on the assigned walk and glue to the considered node two new vertices. If  $\mathfrak{w}_1 = \varepsilon$ , then we let the left vertex as a leaf; otherwise we assign to it the walk  $\mathfrak{w}_1$ . We proceed similarly with the right vertex and  $\mathfrak{w}_2$ . See Figure 4 for an example.

Denoting by  $\Phi_1$  and  $\Phi_2$  the coordinate projections of  $\Phi$  and by  $|\cdot|$  the number of steps in a walk, it is easy to check from the definition of  $\Phi$  that  $|\Phi(\mathfrak{w})| = |\Phi_1(\mathfrak{w})| + |\Phi_2(\mathfrak{w})|$ , when we use the convention that  $|\varepsilon| := 1$ . As a consequence, we see that the number of leaves in the binary tree corresponding to a walk  $\mathfrak{w}$  is  $|\mathfrak{w}|$ . We thus obtain, for each  $n \geq 1$ , a bijection between  $U$ -walks of length  $n + 1$  and binary trees with  $n + 1$  leaves, which are counted by the  $n$ -th Catalan number.

## 2.3 Matched statistics

Let us see now what quantities are matched by our bijection.

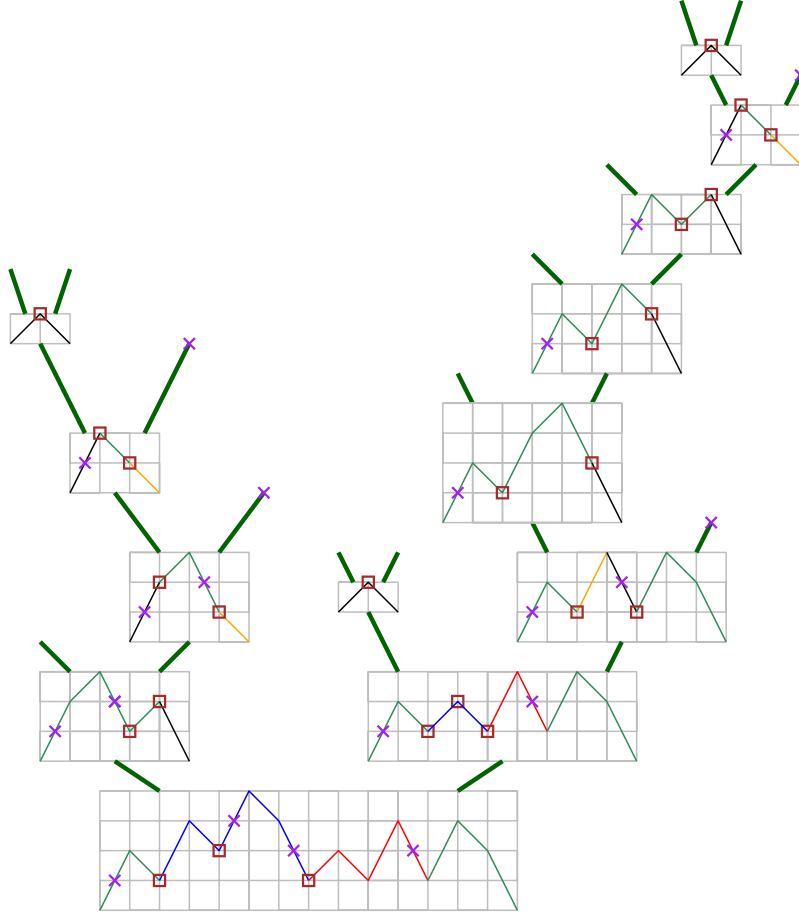
**Proposition 6.** *We consider a  $U$ -walk and a binary tree corresponding through the bijection. Then*

- ◇ *the number of  $\pm 1$ -steps of the walk is equal to the number of double leaves of the tree;*
- ◇ *the sum of the numbers of odd  $+2$ -steps and even  $-2$ -steps of the walk is equal to the number of left leaves of the tree;*
- ◇ *the sum of the numbers of odd  $-2$ -steps and even  $+2$ -steps of the walk is equal to the number of right leaves of the tree.*

*Proof.* We proceed recursively. Let  $d(\mathfrak{w})$  denote the number of  $\pm 1$ -steps of a walk  $\mathfrak{w}$ . As all transformations involved in the definition of  $\Phi$  can only change a  $k$ -step into a  $k$ -step or a  $-k$ -step, we see that, for  $\mathfrak{w} \in \mathcal{U} \setminus \{(0, 1, 0)\}$ , we have  $d(\mathfrak{w}) = d(\Phi_1(\mathfrak{w})) + d(\Phi_2(\mathfrak{w}))$ , with the convention that  $d(\varepsilon) := 0$ . Moreover, the walk  $(+1, -1)$  has two  $\pm 1$ -steps and creates two double leaves, whereas any other walk does not create any double leaves. The first statement of the proposition follows.

Now remember that  $A$ -walks and  $U$ -walks go from 0 to 0 and that  $B$ -walks go from 0 to 1. Moreover, the quantity  $\ell$  of interest in the second item remains unchanged after a shift of even height or after a shift of odd height and a reversal. Bearing this in mind, observe that  $\Phi$  was chosen in such a way that, in class (i),  $\ell(\mathfrak{w}) = 0$ , in class (ii),  $\ell(\mathfrak{w}) = \ell(\Phi_1(\mathfrak{w})) + \ell(\Phi_2(\mathfrak{w}))$ , in class (iii),  $\ell(\mathfrak{w}) = 1 + \ell(\Phi_2(\mathfrak{w}))$ , and in class (iv),  $\ell(\mathfrak{w}) = \ell(\Phi_1(\mathfrak{w}))$ . The second statement follows from the fact that walks of class (iii) create exactly one left leaf whereas walks of other classes do not create any.

The third statement follows by a similar reasoning or by noticing that the sum of the three considered quantities is the length of the walk and the number of leaves in the tree, and we already observed that these quantities were matched by the bijection.  $\square$



**Figure 4:** The bijection between a 14-step U-walk and a 14-leaf binary tree. The original U-walk is represented at the root of the tree and the walks assigned to the nodes of the tree are represented. We used the same color scheme as on the left of Figure 3 and we represented by purple crosses the odd  $+2$ -steps and even  $-2$ -steps of the walks; observe that these are preserved at every stage except when a left leaf is created (we also represented them by purple crosses), in which case one such step disappears. The other  $\pm 2$ -steps satisfy a similar property with right leaves and the  $\pm 1$ -steps correspond to double leaves.

Proposition 1 is a direct consequence of Proposition 6, as the desired number is then equal to the number of binary trees with  $d$  double leaves,  $\ell$  left leaves and  $r$  right leaves. This number can easily be obtained from [MS00, Theorem 4] (which gives the number of non-embedded rooted binary trees with  $n$  labelled leaves, and  $k$  pairs of double leaves, called *cherries*); we provide here a short line of proof for the sake of self-containment (note that one can also obtain the expression through the Lagrange inversion formula, as in the proof of Proposition 3).

*Proof of Proposition 1.* A binary tree with  $2d$  double leaves,  $\ell$  left leaves and  $r$  right leaves can be obtained as follows. We start from a planted binary tree with  $d$  leaves and we add  $\ell + r$  extra vertices on its  $2d - 1$  edges. We graft two new edges to each of the  $d$  original leaves. Among the

added vertices, we select  $\ell$  vertices to the left of which we graft an edge and we graft an edge to the right of the unselected vertices. As this operation is unequivocal, we obtain that the desired number is equal to the  $d-1$ -th Catalan number, times the number of ways to put  $\ell+r$  extra vertices on  $2d-1$  edges, times the number of ways to select  $\ell$  vertices among  $\ell+r$ .  $\square$

## 2.4 Walks without large steps

Let us restrict our attention to  $U$ -walks without  $\pm 2$ -steps. Such walks are in direct bijection with  $A$ -walks without  $\pm 2$ -steps by removing the first and last steps (recall that a  $U$ -walk may not visit 0 more than twice, whereas an  $A$ -walk may) and the latter are exactly Dyck walks. If  $w$  is such a  $U$ -walk, it can only be in the classes (i) or (ii). Moreover, the only  $B$ -walk without  $\pm 2$ -steps is the trivial walk  $(+1)$ , so that, if  $w$  is of class (ii), it is of the form  $(+1, +1)a_1(-1)a_2(-1)$  where  $a_1$  and  $a_2$  are  $A$ -walks without  $\pm 2$ -steps. As a consequence, we see that  $\Phi((+1, +1)a_1(-1)a_2(-1)) = ((+1)\bar{a}_1(-1), (+1)a_2(-1))$  and our bijection is in this case a slight modification of the classical bijection between Dyck walks and binary trees.

Recall that the classical bijection may be constructed in a similar way as ours with a function  $\Psi$  defined on nonempty Dyck walks by  $\Psi((+1)a_1(-1)a_2) := (a_1, a_2)$ . Then, from a Dyck walk, we recursively construct the tree by starting with a vertex tree and assigning to its vertex the walk. Then, we split every vertex with an assigned nonempty walk  $a$  into two new vertices to which we assign the coordinates of  $\Psi(a)$ . Through this bijection, the length of the Dyck walk is twice the number of leaves minus 2.

Plainly, our bijection gives the same result with two double leaves grafted onto each leaf (as the empty Dyck walk yields a leaf in the classical bijection and the corresponding  $U$ -walk  $(+1, -1)$  yields two double leaves in our construction) and where each left subtree is mirrored at each step (because of the reversal of  $a_1$ ).

Summing up, we see that the restriction of our bijection to  $U$ -walks without  $\pm 2$ -steps gives a bijection between such walks and binary trees, which is a slight alteration of the classical encoding of binary trees by Dyck walks. Note that, here, the length of the  $U$ -walk is twice the number of leaves of the corresponding binary tree (equivalently, it is the number of leaves of the tree obtained by adding two double leaves to each original leaf).

## 3 Link with increasing unary-binary trees

### 3.1 Proof of Theorem 2

Let us now investigate the connection between basketball walks and increasing unary-binary trees with associated permutation avoiding 213. We start by giving a more convenient characterization of the permutations allowed on a given unary-binary tree. We say that a permutation  $\sigma = (\sigma_1, \dots, \sigma_n)$  is *valid* for a unary-binary tree if the labeled tree obtained by labeling the vertices by  $\sigma_1, \dots, \sigma_n$  in breadth-first search order is an increasing unary-binary tree with associated permutation avoiding 213. In other words, the labeled tree is increasing and  $\sigma$  avoids 213.

**Lemma 7.** *We consider an  $n$ -vertex unary-binary tree and denote by  $\mathcal{N} \subseteq \{1, 2, \dots, n\}$  the set of indices of its nodes when reading its vertices in breadth-first search order. The permutation  $\sigma$  is valid for the tree if and only if it avoids 213 and, for all  $i \in \mathcal{N}$ ,  $\sigma_i = \min_{i \leq j \leq n} \sigma_j$ ; in other words, the elements of  $\mathcal{N}$  are future record times.*

**Remark.** This observation was made in the very particular case of complete unary-binary trees, for which the vertices have a very simple structure as they are arranged by nodes first and leaves last; see [LPRS16, Theorem 4].

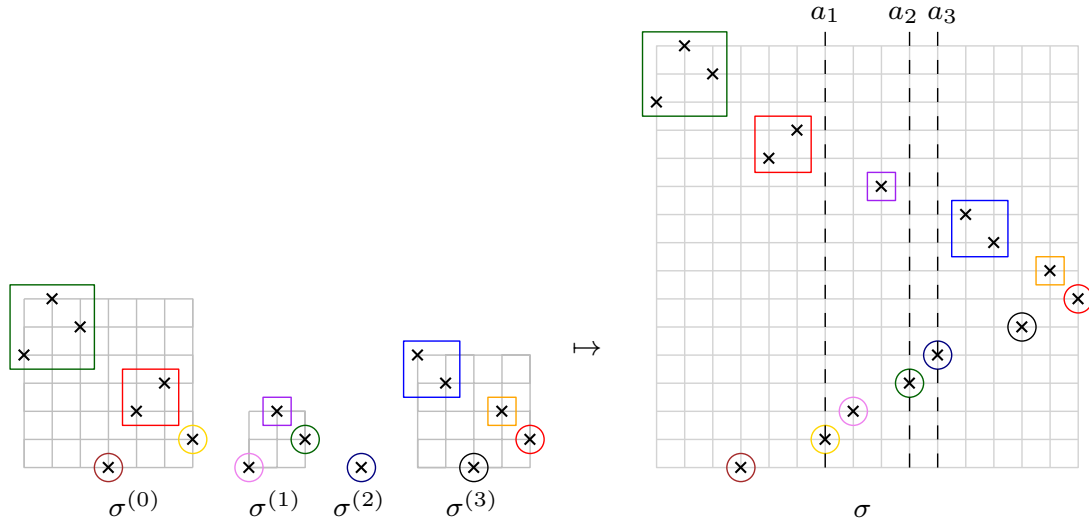
*Proof.* Let us denote by  $v_1, \dots, v_n$  the vertices of the tree read in breadth-first search order. By definition,  $\sigma$  is valid for the tree if and only if it avoids 213 and, for all  $i \in \mathcal{N}$  and all  $j \in \{1, \dots, n\}$  such that  $v_j$  is a child of  $v_i$ , we have  $\sigma_i < \sigma_j$ . By definition of the breadth-first search order, if  $i$  and  $j$  are as above, then  $i < j$ . As a result, a permutation for which the times of  $\mathcal{N}$  are future record times is immediately valid.

Conversely, let us take a valid permutation  $\sigma$  and an index  $i \in \mathcal{N}$ ; it is sufficient to show that  $i$  is a future record time, that is,  $\sigma_i < \sigma_k$  for all  $k \in \{i+1, \dots, n\}$ . Let  $j \in \{1, \dots, n\}$  be the largest integer such that  $v_j$  is a child of  $v_i$ . First,  $\sigma_i < \sigma_j$  as the labeled tree is increasing. Second, we must have  $\sigma_i < \sigma_k$  for all  $k \in \{i+1, \dots, j-1\}$  as  $\sigma_i < \sigma_j$  and  $(\sigma_i, \sigma_k, \sigma_j)$  must avoid 213. Last, if  $k \in \{j+1, \dots, n\}$ , then  $v_k$  must have an ancestor  $v_{k'}$  for some  $k' \in \{i+1, \dots, j\}$  and we see that  $\sigma_k > \sigma_{k'} > \sigma_i$  by the above observation.  $\square$

**Lemma 8.** For  $n \geq 1$ , let  $\mathcal{P}_n$  be the set of 213-avoiding permutations of size  $n$  and, for each nonempty subset  $\mathcal{E} \subseteq \{1, \dots, n-1\}$ , let  $\mathcal{P}_n^{(\mathcal{E})} := \{\sigma \in \mathcal{P}_n : \forall a \in \mathcal{E}, a \text{ is a future record time of } \sigma\}$ . Then, for such a subset  $\mathcal{E} = \{a_1 < \dots < a_k\}$ , we have

$$\mathcal{P}_n^{(\mathcal{E})} \simeq \mathcal{P}_{a_1} \times \mathcal{P}_{a_2 - a_1} \times \dots \times \mathcal{P}_{a_k - a_{k-1}} \times \mathcal{P}_{n - a_k}.$$

*Proof.* The mapping from  $\sigma \in \mathcal{P}_n^{(\mathcal{E})}$  to  $(\sigma^{(0)}, \dots, \sigma^{(k)}) \in \mathcal{P}_{a_1} \times \mathcal{P}_{a_2 - a_1} \times \dots \times \mathcal{P}_{a_k - a_{k-1}} \times \mathcal{P}_{n - a_k}$  is very simple: for  $i \in \{0, \dots, k\}$ , we let  $\sigma^{(i)}$  be the renormalized permutation induced by  $\sigma$  on the index-set  $\{a_i + 1, \dots, a_{i+1}\}$  (with the convention  $a_0 = 0$  and  $a_{k+1} = n$ ).



**Figure 5:** From a  $(k+1)$ -sequence  $(\sigma^{(0)}, \dots, \sigma^{(k)})$  of nonempty 213-avoiding permutations ( $k = 3$  in the example) to a 213-avoiding permutation  $\sigma$  of size  $|\sigma^{(0)}| + \dots + |\sigma^{(k)}|$ . If we let  $a_i := |\sigma^{(0)}| + \dots + |\sigma^{(i-1)}|$  for each  $i \in \{1, \dots, k\}$ , then  $a_i$  is a future record time of  $\sigma$ . On the figure, the future record times are circled and the (nonempty) blocks are framed.

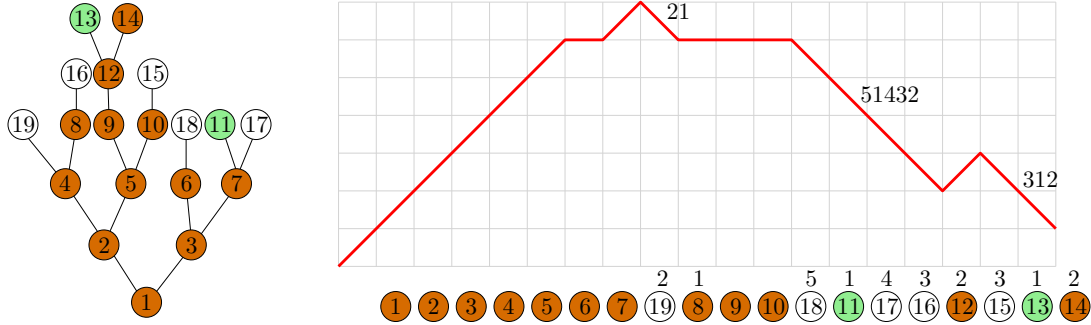
To inverse the mapping we rely on the crucial property that a permutation  $\sigma \in \mathcal{P}_n$  with  $p$  future record times is of the form  $\sigma = (\sigma^{(0)}, 1, \sigma^{(1)}, 2, \sigma^{(2)}, 3, \dots, \sigma^{(p-1)}, p)$  where  $1, \dots, p$  are the values taken at the future record times and  $\sigma^{(0)}, \dots, \sigma^{(p-1)}$  are (possibly empty) subpermutations of  $\{p+1, \dots, n\}$  avoiding 213 with disjoint ranges arranged in decreasing order (that is,  $\min \sigma^{(i)} > \max \sigma^{(j)}$  whenever  $i < j$  and  $\sigma^{(i)}, \sigma^{(j)} \neq \emptyset$ ); these subpermutations are called the *blocks* of  $\sigma$ .

Starting from  $(\sigma^{(0)}, \dots, \sigma^{(k)}) \in \mathcal{P}_{a_1} \times \mathcal{P}_{a_2-a_1} \times \dots \times \mathcal{P}_{a_k-a_{k-1}} \times \mathcal{P}_{n-a_k}$ , we get to  $\sigma \in \mathcal{P}_n^{(\mathcal{E})}$  as follows: we concatenate  $\sigma^{(0)}, \dots, \sigma^{(k)}$  and renormalize the values in the unique way so that the future record times of  $\sigma$  are those of  $\sigma^{(0)}, \dots, \sigma^{(k)}$  in increasing order (with consecutive values) and the blocks of  $\sigma$  are those of  $\sigma^{(0)}, \dots, \sigma^{(k)}$ , with ranges in decreasing order; see Figure 5 for an example. Note that, for each  $i \in \{0, \dots, k-1\}$ , since the last index in  $\sigma^{(i)}$  is necessarily a future record time for  $\sigma^{(i)}$ , the corresponding index  $a_{i+1}$  is a future record time for  $\sigma$ .  $\square$

Using the previous two lemmas, we may now give a more convenient encoding of unary-binary trees with associated permutation avoiding 213. Recall that a *Motzkin walk* is a walk with step-set  $\{-1, 0, +1\}$ . We call *descending sequence* of a Motzkin walk a maximal run of  $-1$ 's, that is, a sequence of consecutive steps of the form  $(-1, \dots, -1)$  that is either initial or preceded by a  $+1$ - or  $0$ -step and is either final or succeeded by a  $+1$ - or  $0$ -step. We finally say that a Motzkin walk is *decorated* if each of its descending sequences receives a permutation avoiding 213 whose size is the length of the descending sequence (its number of  $-1$ ) plus one.

**Lemma 9.** For  $n \geq 1$ ,  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 are bijectively encoded by  $n$ -step decorated Motzkin walks from 0 to 1 that are positive except at the origin.

Figure 6 gives an example of this encoding.



**Figure 6:** Left: an increasing unary-binary tree with associated permutation avoiding 213. Right: its encoding as a decorated Motzkin walk. The nodes and the last leaf are colored in brown and the other future record times are colored in green.

*Proof.* Let  $T$  be a unary-binary tree with  $n$  vertices and associated permutation  $\sigma$  avoiding the pattern 213. As above, we denote by  $v_1, \dots, v_n$  the vertices of  $T$  in the (left to right) breadth-first search order. We classically encode  $T$  as a Motzkin walk as follows: if we denote by  $c(v) \in \{0, 1, 2\}$  the number of children of the vertex  $v$ , the walk  $w$  associated with  $T$  starts at zero and follows the

step sequence  $(+1, c(v_1) - 1, c(v_2) - 1, \dots, c(v_{n-1}) - 1)$ . The usual description<sup>4</sup> follows a slightly different step sequence; namely the first  $+1$  is omitted and a final step  $c(v_n) - 1 = -1$  is added. This only changes the fact that we obtain a positive walk from 0 to 1 instead of a nonnegative walk from 0 to  $-1$ .

Let us now take into account the labels of the tree. Let  $\mathcal{N} = \{a_1 < \dots < a_k\}$  be the set of indices of the nodes of  $T$  (note that  $a_1 = 1$ ). According to Lemma 7, each  $a \in \mathcal{N}$  is a future record time of  $\sigma$ . Hence,  $\sigma \in \mathcal{P}_n^{(\mathcal{N})}$  and, by Lemma 8,  $\sigma$  identifies to the  $(k+1)$ -tuple  $(\sigma^{(0)}, \dots, \sigma^{(k)})$ , where  $\sigma^{(i)}$  is the renormalized permutation induced by  $\sigma$  on the index-set  $\{a_i + 1, \dots, a_{i+1}\}$  (with the convention  $a_0 = 0$  and  $a_{k+1} = n$ ).

Plainly, in the Motzkin walk  $\mathfrak{w}$ , the initial  $+1$ -step corresponds to the last leaf of  $T$ , the other  $+1$ - and  $0$ -steps correspond to the nodes of  $T$  and the  $-1$ -steps correspond to the leaves of  $T$  except for the last one. This entails that, for each  $i \in \{0, \dots, k\}$  such that  $a_{i+1} - a_i \geq 2$ , the steps of  $\mathfrak{w}$  between indices  $a_i + 1$  and  $a_{i+1} - 1$  form a descending sequence of length  $a_{i+1} - a_i - 1$ . To the descending sequence corresponding to such an  $i$ , we assign the permutation  $\sigma^{(i)}$  (of size  $a_{i+1} - a_i$ ). We thus obtain a decorated Motzkin walk that encodes the tree  $T$  with no loss of information, as  $\sigma^{(i)} = (1)$  whenever  $a_{i+1} - a_i = 1$ . See Figure 6 for an example.  $\square$

We now introduce two new generating functions, counted with weight  $z$  per step:

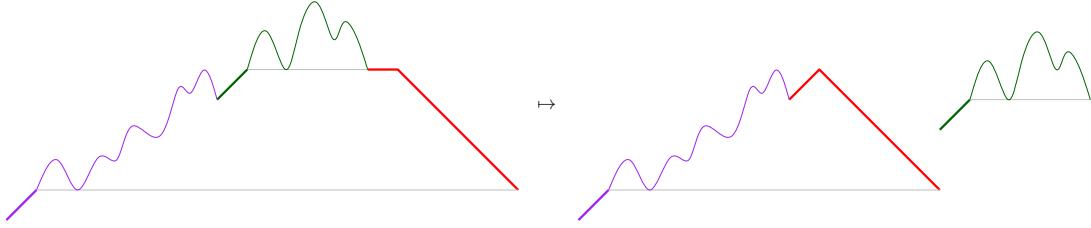
- ◊  $T$  counts decorated Motzkin walks from 0 to 1 that are positive except at the origin;
- ◊  $M$  counts decorated Motzkin walks from 0 to 1 that are positive except at the origin and whose last  $0$ - or  $+1$ -step is a  $+1$ -step.

*Proof of Theorem 2.* In order to conclude, it suffices to show that  $T = G$ . We will first show that  $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$  and then that  $M = zA$ . This will be sufficient as we already noticed in Equation (3) that  $\mathcal{G} = \text{Seq}_{\geq 1}(\mathcal{Z}\mathcal{A})$ .

**First step:** We claim that  $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$ . Indeed, a  $T$ -walk is either an  $M$ -walk or a  $T$ -walk whose last  $0$ - or  $+1$ -step is a  $0$ -step. In the latter case, let us denote by  $k \geq 0$  the number of  $-1$ -steps after the last  $0$ -step (which is necessarily at height  $k+1$ ), and by  $n$  the length of the walk. We consider the last time the walk hits height  $k$  before the last  $0$ -step and we split the walk into three parts at this time and at time  $n - k - 1$ . See Figure 7. We obtain first a decorated Motzkin walk from 0 to  $k$  that stays positive except at its origin, then a  $T$ -walk and finally a decorated walk of the form  $(0, -1, \dots, -1)$  with  $k+1$  steps. We change the  $0$ -step of the last part into a  $+1$ -step and we concatenate the first part with it in order to obtain an  $M$ -walk. Note that this construction also works for  $k = 0$ , in which case the first part is the empty walk and the outcome of the above operation is the pair consisting of the original walk with last step removed, together with the trivial  $M$ -walk  $(+1)$ . As a result, we see that a  $T$ -walk whose last  $0$ - or  $+1$ -step is a  $0$ -step can be bijectively decomposed into a pair made of a  $T$ -walk and an  $M$ -walk. This yields  $\mathcal{T} = \mathcal{M} + \mathcal{T}\mathcal{M}$ , so that  $\mathcal{T} = \text{Seq}_{\geq 1}(\mathcal{M})$  as claimed.

**Second step:** Let us now show that  $M = zA$ . Recall from Section 2.2 the definition of  $A$  and  $B$ . Recall from the discussion before (3) that  $B = z/(1 - zA)$  and from (4) that  $A = 1 + (zA)^2 + (BA)^2$ .

<sup>4</sup>Note that there is also a classical encoding using the depth-first search order instead of the breadth-first search order. Both constructions give bijections between the same sets and the proofs are similar.



**Figure 7:** Proof that  $T$ -walks whose last 0- or +1-step is a 0-step are counted by  $TM$ . We extract a  $T$ -walk (in green) and what remains makes an  $M$ -walk by changing the last 0-step into a +1-step.

As a result,

$$zA = z \left( 1 + (zA)^2 + \frac{(zA)^2}{(1 - zA)^2} \right).$$

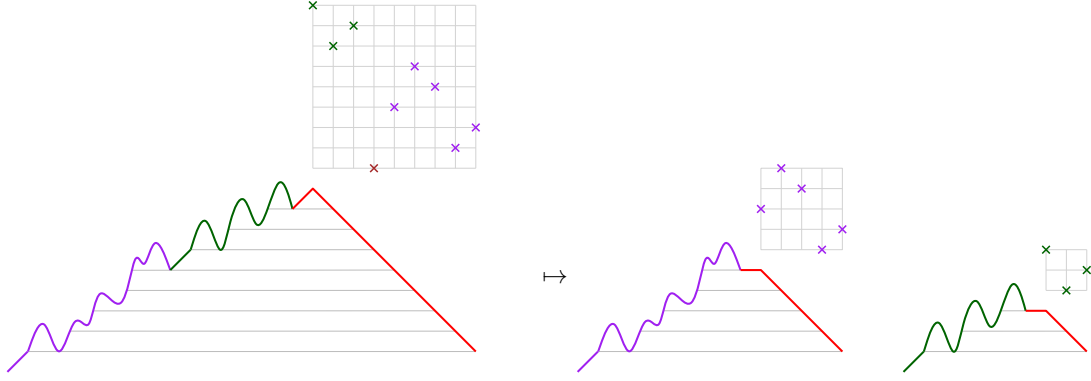
We now turn to  $M$ . We consider an  $M$ -walk and, for the sake of convenience, we suppose for the time being that it is not the trivial walk (+1). Let  $k \geq 1$  denote the length of its last descending sequence and  $\sigma$  denote the permutation of size  $k + 1$  associated with it. We use the standard Catalan decomposition and split  $\sigma$  into two permutations avoiding 213,  $\sigma'$  and  $\sigma''$ , of respective sizes  $i$  and  $k - i$ . This means that, if  $j$  denotes the first future record time of  $\sigma$ , then  $\sigma'$  and  $\sigma''$  correspond to  $(\sigma_1, \dots, \sigma_{j-1})$  and  $(\sigma_{j+1}, \dots, \sigma_{k+1})$ , up to relabeling. Let us suppose furthermore that  $1 \leq i \leq k - 1$  for the moment. We split the  $M$ -walk at the last time it hits  $i$  before the last +1-step and at time  $n - k - 1$  (that is, before the last +1-step); see Figure 8. We concatenate the first part with the  $i$ -sequence  $(0, -1, \dots, -1)$  and we associate with the last descending sequence the permutation  $\sigma'$ . Similarly, we add to the second part a 0-step and a descending sequence of size  $k - i - 1$  with which we associate the permutation  $\sigma''$ . We thus decomposed our original walk into a single step counted by  $z$  and a pair of  $T$ -walks whose last 0- or +1-steps are both 0-steps, which we saw in the first step above are counted by  $TM$ . Now if  $i = k$ , we do the same construction for the first walk and obtain an empty second walk. If  $i = 0$ , we obtain an empty first walk and a second  $TM$ -walk. Finally, if the original walk is trivial, it is counted by  $z$ . Putting all this together, we obtain  $M = z(1 + TM)^2$ , which can be rewritten as

$$M = z \left( 1 + \frac{M^2}{1 - M} \right)^2 = z \left( 1 - M + \frac{M}{1 - M} \right)^2 = z \left( 1 + M^2 + \frac{M^2}{(1 - M)^2} \right).$$

Since  $zA$  and  $M$  satisfy the same Lagrangian equation of the form  $X = z\varphi(X)$ , they must be equal. This concludes the proof.  $\square$

**Note.** Using more thorough decompositions, we will show in Section 3.2 that the identity  $M = zA$  holds in the more general setting of bivariate generating functions, with an extra parameter counting 0-steps in  $M$  and staggered  $\pm 2$ -steps in  $zA$ . This will provide an alternate yet slightly longer proof of the second step above.

We end this section with the following observation. We just showed that  $T = G$ , so that (1) also reads  $1 + T + T^2 = C$ . Let us give an interpretation to this identity.



**Figure 8:** Recursive decomposition of an  $M$ -walk. We split the permutation associated with the last descending sequence into two permutations by the standard Catalan decomposition and we split the walk at the heights corresponding to the sizes of the sub-permutations.

**Proposition 10.** For any  $n \geq 1$ , the  $(n - 1)$ -th Catalan number counts the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 that satisfy the following property: if we let  $\ell_1, \dots, \ell_k$  denote the leaves read after the last node in breadth-first search order, then the last leaf has the smallest label among the labels of  $\ell_1, \dots, \ell_k$ .

*Proof.* Let us consider the decorated Motzkin walk of a tree satisfying the condition of the statement. We consider its last step.

- ◊ If it is a  $+1$ -step, then the walk is the trivial walk  $(+1)$  and the corresponding tree is the vertex-tree with a leaf labeled 1.
- ◊ If it is a  $0$ -step, then the decorated Motzkin walk does not satisfy any further constraints (it is just a decorated Motzkin walk to which a final  $0$ -step is added); the corresponding tree ends with a single leaf after a unary node.
- ◊ If it is a  $-1$ -step, we let  $\sigma = (\sigma_1, \dots, \sigma_k)$  be the permutation assigned to the last descending sequence. The condition of the statement merely translates into  $\sigma_k = 1$ . By considering the last hitting time of 1 before the end, the Motzkin walk can be decomposed into two successive Motzkin walks and a final  $-1$ -step. These Motzkin walks naturally inherit a decoration from the original Motzkin walk, with the exception of the last descending sequence of the second Motzkin walk, to which we assign the permutation  $(\sigma_1 - 1, \dots, \sigma_{k-1} - 1)$ .

The previous classification shows that the generating function of the objects we consider is  $z + zT + zT^2 = zC$ , as desired.  $\square$

### 3.2 Unary nodes and staggered steps

In this section, we focus on the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 that have a given number of unary nodes. We prove Proposition 3 and then identify the statistics on the basketball walks that corresponds to the number of unary nodes in an increasing unary-binary tree. Throughout this section, we will consider bivariate generating



function of walks, with a weight  $z$  per step as above and with an extra weight  $u$  for some steps that will be defined later on. For the sake of clarity, we still denote by  $A$ ,  $B$ ,  $G$ ,  $M$  and  $T$  these bivariate functions.

*Proof of Proposition 3.* Recall that unary nodes of a unary-binary tree correspond to 0-steps of the encoding Motzkin walk. By this observation and Lemma 9, the number of  $n$ -vertex increasing unary-binary trees with associated permutation avoiding 213 having exactly  $n - 1 - 2k$  unary nodes is equal to  $[u^{n-1-2k} z^n] T(z, u)$ , where the variable  $u$  marks the number of 0-steps in the Motzkin walks.

Redoing the proof of Theorem 2 with bivariate functions counting decorated Motzkin walks with an extra weight  $u$  per 0-step, we obtain  $T = M + uTM$  and  $M = z(1 + TM)^2$ , hence  $T = \psi(M)$  and  $M = z\phi(M)$ , where

$$\psi(y) := \frac{y}{1 - uy} \quad \text{and} \quad \phi(y) := \left(1 + \frac{y^2}{1 - uy}\right)^2. \quad (5)$$

Applying Lagrange inversion formula, we obtain

$$\begin{aligned} [z^n] T(z, u) &= \frac{1}{n} [y^{n-1}] \psi'(y) \phi(y)^n = \frac{1}{n} [y^{n-1}] \frac{1}{(1 - uy)^2} \left(1 + \frac{y^2}{1 - uy}\right)^{2n} \\ &= \frac{1}{n} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{k} [y^{n-1-2k}] \frac{1}{(1 - uy)^{k+2}} \\ &= \frac{1}{n} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{2n}{k} \binom{n-k}{k+1} u^{n-1-2k} \end{aligned}$$

and the result follows.  $\square$

We now turn to the proof of Proposition 4. Recall Definition 1 and that  $\mathcal{A}$  is the class of non-negative basketball walks from 0 to 0.

*Proof of Proposition 4.* We denote by  $A(z, u)$  the bivariate generating function of  $\mathcal{A}$ , where  $z$  marks the number of steps and  $u$  the number of staggered  $\pm 2$ -steps. A walk in  $\mathcal{A}$  is

- (i) either empty;
- (ii) or is the concatenation of a  $+2$ -step, an  $A$ -walk, a  $-2$ -step and an  $A$ -walk;
- (iii) or visits 1 (at least once) before its first return to 0. In that latter case, the walk is the concatenation of a  $B$ -walk, an  $A$ -walk, a reversed  $B$ -walk and an  $A$ -walk. We distinguish three subcases:
  - (a) the two  $B$ -walks are trivial, that is, equal to  $(+1)$ ,
  - (b) exactly one of the two  $B$ -walks is trivial,
  - (c) both  $B$ -walks have length 2 or more.

Now observe that, in case (i), (ii), (iii)(a) and (iii)(b), each staggered  $\pm 2$ -step of the original walk remains a staggered  $\pm 2$ -step in one of its subwalks. In case (iii)(c), both  $B$ -walks are composed of a first  $+2$ -step that is not staggered in the original walk, then an  $A$ -walk and a reversed  $B$ -walk. The previous decomposition thus gives the following identity in terms of bivariate generating functions:

$$A = \underbrace{1}_{(i)} + \underbrace{(zA)^2}_{(ii)} + \underbrace{(zA)^2}_{(iii)(a)} + \underbrace{2zA^2(B-z)}_{(iii)(b)} + \underbrace{(zAB)^2A^2}_{(iii)(c)}.$$

Let us express  $B$  in terms of  $A$ . Following the discussion that led to (3), a  $B$ -walk is either the trivial walk  $(+1)$ , or a staggered  $+2$ -step followed by an  $A$ -walk then a reversed  $B$ -walk. As a result,  $B = z + uzAB$ , so that  $B = z/(1 - uzA)$ . We thus obtain

$$\begin{aligned} zA &= z \left( 1 + 2(zA)^2 + 2u \frac{(zA)^3}{1 - uzA} + \frac{(zA)^4}{(1 - uzA)^2} \right) \\ &= z \left( 1 + \frac{2(zA)^2}{1 - uzA} + \frac{(zA)^4}{(1 - uzA)^2} \right) = z \left( 1 + \frac{(zA)^2}{1 - uzA} \right)^2, \end{aligned}$$

which is also the Lagrangian equation (5) verified by  $M$ . In order to conclude the proof, it is enough to note that  $G = AB = zA/(1 - uzA)$ , and that  $T = M/(1 - uM)$ .  $\square$

*Proof of Proposition 5.* We obtained during the proof of Proposition 4 that  $zA = z\phi(zA)$  where  $\phi$  is defined by (5). By the Lagrange inversion formula, we have, for  $n \geq 2$ ,

$$\begin{aligned} [z^{n-1}]A(z, u) &= [z^n]zA(z, u) = \frac{1}{n} [y^{n-1}]\phi(y)^n = \frac{1}{n} [y^{n-1}] \left( 1 + \frac{y^2}{1 - uy} \right)^{2n} \\ &= \frac{1}{n} \sum_{k \geq 0} \binom{2n}{k} [y^{n-1-2k}] \frac{1}{(1 - uy)^k} \\ &= \frac{1}{n} \sum_{k \geq 0} \binom{2n}{k} \binom{n-k-2}{k-1} u^{n-1-2k} \end{aligned}$$

and the result follows. Note that the formula  $[y^a](1 - y)^{-b} = \binom{a+b-1}{b-1}$  is not valid for  $a = 0$  and  $b = 0$ ; this is why we assumed  $n \geq 2$ .  $\square$

### 3.3 Turning Theorem 2 into an explicit bijection

The point of Section 2.2 was to write the grammar of basketball walks and that of binary trees in an isomorphic way. Namely, by thoroughly decomposing  $U$ -walks, we manage to write  $\mathcal{U} = \mathcal{Z}^2 + 2\mathcal{Z}\mathcal{U} + \mathcal{U}^2$ , and the same can be done trivially for the grammar of binary trees. We could then recursively write an object of the first grammar as an object of the second grammar.

We can do the same thing in order to obtain an explicit bijection between  $G$ -walks and increasing unary-binary trees with associated permutation avoiding 213. In fact, we have already done in the previous sections all the necessary operations. Moreover, the study we did with bivariate generating functions in Section 3.2 will yield that the statistics counted by  $u$  in  $\mathcal{G}$  will correspond to the statistics counted by  $u$  in  $\mathcal{T}$ .

Recall Equation (3) and the proof of Theorem 2. We showed that  $\mathcal{G} = \mathcal{ZA} + \mathcal{ZAG}$  and that  $\mathcal{T} = \mathcal{M} + \mathcal{MT}$  in a combinatorially tractable way. We mean by this that we can explicitly write an object of  $\mathcal{G}$  as either an object of  $\mathcal{ZA}$  or a pair of an object of  $\mathcal{ZA}$  and one of  $\mathcal{G}$  and the same holds with  $\mathcal{M}$  instead of  $\mathcal{ZA}$  and  $\mathcal{T}$  instead of  $\mathcal{G}$ . Moreover, these equations hold indifferently for univariate or bivariate generating functions. As a result, it suffices to explain how  $\mathcal{ZA}$  and  $\mathcal{M}$  can be decomposed in an isomorphic way. But this was actually done in Section 3.2. Indeed, we combinatorially obtained

$$\mathcal{ZA} = \mathcal{Z} \left( \mathbf{1} + 2(\mathcal{ZA})^2 + 2u(\mathcal{ZA})^3 \text{Seq}(u\mathcal{ZA}) + (\mathcal{ZA})^4 (\text{Seq}(u\mathcal{ZA}))^2 \right),$$

as well as

$$\mathcal{M} = \mathcal{Z} \left( \mathbf{1} + \mathcal{M}^2 \text{Seq}(u\mathcal{M}) \right)^2.$$

The latter equation can be rewritten as the first one by developing the square, which, combinatorially, means distinguishing whether each element of the pair is empty or not and by using the identity  $\text{Seq}(u\mathcal{M}) = \mathbf{1} + u\mathcal{M} \text{Seq}(u\mathcal{M})$ , which translates into saying that a sequence is empty or is composed of a first element and another sequence.

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